THE INTEGRALS IN GRADSHTEYN AND RHYZIK. PART 2: ELEMENTARY LOGARITHMIC INTEGRALS

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ABSTRACT. We describe methods to evaluate elementary logarithmic integrals. The integrand is the product of a rational function and a linear polynomial in $\ln x$.

1. INTRODUCTION

The table of integrals by I. M. Gradshteyn and I. M. Rhyzik [3] contains a large selection of definite integrals of the form

$$(1.1) \qquad \int_a^b R(x) \ln^m x \, dx,$$

where $R(x)$ is a rational function, $a, b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. We call integrals of the form (1.1) *elementary logarithmic integrals*. The goal of this note is to present methods to evaluate them. We may assume that $a = 0$ using

$$(1.2) \qquad \int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \ln^m x \, dx - \int_0^a R(x) \ln^m x \, dx.$$

Section 2 describes the situation when R is a polynomial. Section 3 presents the case in which the rational function has a single simple pole. Finally section 4 considers the case of multiple poles.

2. POLYNOMIALS EXAMPLES

The first example considered here is

$$(2.1) \qquad I(P; b, m) := \int_0^b P(x) \ln^m x \, dx,$$

where P is a polynomial. This can be evaluated in elementary terms. Indeed, $I(P; b, m)$ is a linear combination of

$$(2.2) \qquad \int_0^b x^j \ln^m x \, dx,$$

and the change of variables $x = bt$ yields

$$(2.3) \qquad \int_0^b x^j \ln^m x \, dx = b^{j+1} \sum_{k=0}^m \binom{m}{k} \ln^{m-k} b \int_0^1 t^j \ln^k t \, dt.$$

The last integral evaluates to $(-1)^k k!/(j+1)^{k+1}$ either an easy induction argument or by the change of variables $t = e^{-s}$ that gives it as a value of the gamma function.

Theorem 2.1. Let $P(x)$ be a polynomial given by

$$(2.4) \qquad P(x) = \sum_{j=0}^{p} a_j x^j.$$

Then
(2.5)

$$I(P; b, m) := \int_0^b P(x) \ln^m x\, dx = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} \ln^{m-k} b \sum_{j=0}^{p} a_j \frac{b^{j+1}}{(j+1)^{k+1}}.$$

This expression shows that $I(P; b, m)$ is a linear combination of $b^j \ln^k b$, with $1 \le j \le 1 + p(= 1 + \deg(P))$ and $0 \le k \le m$.

3. Linear denominators

We now consider the integral

$$(3.1) \qquad f(b; r) := \int_0^b \frac{\ln x\, dx}{x + r}$$

for $b, r > 0$. This corresponds to the case in which the rational function in (1.1) has a single simple pole.

The change of variables $x = rt$ produces

$$(3.2) \qquad \int_0^b \frac{\ln x\, dx}{x + r} = \ln r\, \ln(1 + b/r) + \int_0^{b/r} \frac{\ln t\, dt}{1 + t}.$$

Therefore, it suffices to consider the function

$$(3.3) \qquad g(b) := \int_0^b \frac{\ln t\, dt}{1 + t},$$

as we have

$$(3.4) \qquad f(b; r) = \ln r\, \ln\left(1 + \frac{b}{r}\right) + g\left(\frac{b}{r}\right).$$

Before we present a discussion of the function g, we describe some elementary consequences of (3.2).

Elementary examples. The special case $r = b$ in (3.2) yields

$$(3.5) \qquad \int_0^b \frac{dx}{x + b} = \ln 2\, \ln b + \int_0^1 \frac{\ln t\, dt}{1 + t}.$$

Expanding $1/(1 + t)$ as a geometric series, we obtain

$$(3.6) \qquad \int_0^1 \frac{\ln t\, dt}{1 + t} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}.$$

This appears as **4.231.1** in [3]. Differentiating (3.2) with respect to r produces

$$(3.7) \qquad \int_0^b \frac{\ln x\, dx}{(x + r)^2} = -\frac{\ln(b + r)}{r} + \frac{\ln r}{r} + \frac{b \ln b}{r(r + b)}.$$

As b, $r \to 1$ we obtain

$$(3.8) \qquad \int_0^1 \frac{\ln x\,dx}{(1+x)^2} = -\ln 2.$$

This appears as **4.231.6** in [3]. On the other hand, as $b \to \infty$ we recover **4.231.5** in [3]:

$$(3.9) \qquad \int_0^\infty \frac{\ln x\,dx}{(x+r)^2} = \frac{\ln r}{r}.$$

The polylogarithm function. The evaluation of the integral

$$(3.10) \qquad g(b) := \int_0^b \frac{\ln t\,dt}{1+t},$$

requires the transcendental function

$$(3.11) \qquad \mathrm{Li}_n(x) := \sum_{k=1}^\infty \frac{x^k}{k^n}.$$

This is the *polylogarithm function* and it has also appeared in [5] in our discussion of the family

$$(3.12) \qquad h_n(a) := \int_0^\infty \frac{\ln^n x\,dx}{(x-1)(x+a)}, \quad n \in \mathbb{R},\ a > 0.$$

In the current context we have $n = 2$ and we are dealing with the *dilogarithm function*: $\mathrm{Li}_2(x)$.

Lemma 3.1. The function $g(b)$ is given by

$$(3.13) \qquad g(b) = \ln b \ln(1+b) + \mathrm{Li}_2(-b).$$

Proof. The change of variables $t = bs$ yields

$$(3.14) \qquad g(b) = \ln b \ln(1+b) + \int_0^1 \frac{\ln s\,ds}{1+bs}.$$

Expanding the integrand in a geometric series yields the final identity. $\qquad\square$

Theorem 3.2. Let b, $r > 0$. Then

$$(3.15) \qquad \int_0^b \frac{\ln x\,dx}{x+r} = \ln b \ln\left(\frac{b+r}{r}\right) + \mathrm{Li}_2\left(-\frac{b}{r}\right).$$

Corollary 3.3. Let $b > 0$. Then

$$(3.16) \qquad \int_0^b \frac{\ln x\,dx}{x+b} = \ln 2 \ln b - \frac{\pi^2}{12}.$$

Proof. Let $r \to b$ in Theorem 3.2 and use

$$(3.17) \qquad \mathrm{Li}_2(-1) = \sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

$$\square$$

The expression in Theorem 3.2 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:

Corollary 3.4. Let $0 < a < b$ and $r_1 \neq r_2 \in \mathbb{R}^+$. Then, with $r = r_2 - r_1$, we have

$$\int_a^b \frac{\ln x \, dx}{(x + r_1)(x + r_2)} = \frac{1}{r}\left[\ln b \ln\left(\frac{r_2(b + r_1)}{r_1(b + r_2)}\right) + \ln a \ln\left(\frac{r_1(a + r_2)}{r_2(a + r_1)}\right)\right] +$$

$$+ \frac{1}{r}\left[\mathrm{Li}_2\left(-\frac{b}{r_1}\right) - \mathrm{Li}_2\left(-\frac{a}{r_1}\right) - \mathrm{Li}_2\left(-\frac{b}{r_2}\right) + \mathrm{Li}_2\left(-\frac{a}{r_2}\right)\right].$$

The special case $a = r_1$ and $b = r_2$ is of interest:

Corollary 3.5. Let $0 < a < b$. Then

$$\int_a^b \frac{\ln x \, dx}{(x + a)(x + b)} = \frac{1}{b - a}\left[\ln(ab)\ln(a + b) - \ln 2 \ln(ab) - 2 \ln a \ln b\right]$$

$$+ \frac{1}{b - a}\left[-2\mathrm{Li}_2(-1) + \mathrm{Li}_2\left(-\frac{b}{a}\right) + \mathrm{Li}_2\left(-\frac{a}{b}\right)\right].$$

The integral in Corollary 3.5 appears as **4.232**.1 in [3]. An interesting problem is to derive **4.232**.2

$$(3.18) \qquad \int_0^\infty \frac{\ln x \, dx}{(x + u)(x + v)} = \frac{\ln^2 u - \ln^2 v}{2(u - v)}$$

directly from Corollary 3.5.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that

$$(3.19) \qquad \int_a^b \frac{\ln x \, dx}{(x + a)(x + b)} = \frac{\ln ab}{2(b - a)} \ln \frac{(a + b)^2}{4ab}.$$

Proof. The partial fraction decomposition

$$\frac{1}{(x + a)(x + b)} = \frac{1}{b - a}\left(\frac{1}{x + a} - \frac{1}{x + b}\right).$$

reduces the problem to the evaluation of

$$I_1 = \int_a^b \frac{\ln x \, dx}{x + a} \text{ and } I_2 = \int_a^b \frac{\ln x \, dx}{x + b}.$$

The change of variables $x = at$ gives, with $c = b/a$,

$$I_1 = \int_1^c \frac{\ln(at) \, dt}{1 + t}$$

$$= \ln a \int_1^c \frac{dt}{1 + t} + \int_1^c \frac{\ln t}{1 + t} dt$$

$$= \ln a \ln(1 + c) - \ln a \ln 2 + \int_1^c \frac{\ln t}{1 + t} dt.$$

Similarly,

$$I_2 = \ln b \ln 2 - \ln b \ln(1 + 1/c) + \int_1^{1/c} \frac{\ln t}{1 + t} dt.$$

Therefore

$$I_1 - I_2 = \ln a \ln(1+c) + \ln b \ln(1+1/c) - \ln 2 \ln a - \ln 2 \ln b +$$
$$+ \int_1^c \frac{\ln t}{1+t} dt - \int_{1/c}^1 \frac{\ln t}{1+t} dt.$$

Let $s = 1/t$ in the second integral to get

$$\int_{1/c}^1 \frac{\ln t}{1+t} dt = \int_c^1 \frac{\ln s}{s(1+s)} ds.$$

Replacing in the expression for $I_1 - I_2$ yields

$$I_1 - I_2 = \ln a \left(\ln(a+b) - \ln a - \ln 2 \right) - \ln b \left(\ln 2 - \ln(a+b) + \ln b \right) +$$
$$+ \int_1^c \frac{\ln t}{t} dt.$$

The last integral can now be evaluated by elementary means to produced the result.
□

Now comparing the two evaluation of the integral in Corollary 3.5 produces an identity for the dilogarithm function.

Corollary 3.6. The dilogarithm function satisfies

$$(3.20) \qquad \mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(z).$$

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [4] a nice description of them.

4. A SINGLE MULTIPLE POLE

In this section we consider the evaluation of

$$(4.1) \qquad f_n(b, r) := \int_0^b \frac{\ln x \, dx}{(x+r)^n}.$$

This corresponds to the elementary rational integrals with a single pole (at $x = -r$). The change of variables $x = rt$ yields

$$f_n(b, r) = \frac{\ln r}{(n-1)r^{n-1}} \left[\frac{(b+r)^{n-1} - r^{n-1}}{(b+r)^{n-1}} \right] + \frac{1}{r^{n-1}} h_n(b/r),$$

where

$$(4.2) \qquad h_n(b) := \int_0^b \frac{\ln t \, dt}{(1+t)^n}.$$

We first establish a recurrence for h_n.

Theorem 4.1. Let $n > 2$ and $b > 0$. Then h_n satisfies the recurrence

$$(4.3) \quad h_n(b) = \frac{n-2}{n-1} h_{n-1}(b) + \frac{b \ln b}{(n-1)(1+b)^{n-1}} + \frac{1 - (1+b)^{n-2}}{(n-1)(n-2)(1+b)^{n-2}}.$$

Proof. Start with

$$h_n(b) = \int_0^b \frac{[(1+t) - t] \ln t \, dt}{(1+t)^n} = h_{n-1}(b) - \int_0^b \frac{t \ln t \, dt}{(1+t)^n}.$$

Integrate by parts in the last integral, with $u = t \ln t$ and $dv = dt/(1+t)^n$ to produce the result. $\qquad\square$

The initial condition for this recurrence is obtained from the value

(4.4)
$$h_2(b) = \frac{b}{1+b} \ln b - \ln(1+b).$$

This expression follows by a direct integration by parts in

(4.5)
$$h_2(b) = -\lim_{\epsilon \to 0} \int_\epsilon^b \ln t \, \frac{d}{dt}(1+t)^{-1} \, dt.$$

The first few values of $h_n(b)$ suggest the introduction of the function

(4.6)
$$q_n(b) := (1+b)^{n-1} h_n(b),$$

for $n \geq 2$. For example,

(4.7)
$$q_2(b) = b \ln b - (1+b) \ln(1+b).$$

The recurrence for h_n yields one for q_n.

Corollary 4.2. The recurrence

(4.8)
$$q_n(b) = \frac{(n-2)}{(n-1)}(1+b)q_{n-1}(b) + \frac{b \ln b}{n-1} - \frac{(1+b)\left[(1+b)^{n-2} - 1\right]}{(n-1)(n-2)},$$

holds for $n \geq 2$.

Corollary 4.2 establishes the existence of functions $X_n(b)$, $Y_n(b)$ and $Z_n(b)$, such that

(4.9)
$$q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1+b) + Z_n(b).$$

The recurrence (4.8) produces explicit expression for each of these parts.

Proposition 4.3. Let $n \geq 2$ and $b > 0$. Then

(4.10)
$$X_n(b) = \frac{(1+b)^{n-1} - 1}{n-1}.$$

Proof. The function X_n satisfies the recurrence

(4.11)
$$X_n(b) = \frac{n-2}{n-1}(1+b)X_{n-1}(b) + \frac{b}{n-1}.$$

The initial condition is $X_2(b) = b$. The result is now easily established by induction. $\qquad\square$

Proposition 4.4. Let $n \geq 2$ and $b > 0$. Then

(4.12)
$$Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}.$$

Proof. The function Y_n satisfies the recurrence

(4.13)
$$Y_n(b) = \frac{n-2}{n-1}(1+b)Y_{n-1}(b).$$

This recurrence and the initial condition $Y_2(b) = -(1+b)$, yield the result. $\qquad\square$

It remains to identify the function $Z_n(b)$. It satisfies the recurrence

$$(4.14) \qquad Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)\left[(1+b)^{n-2} - 1\right]}{(n-2)(n-1)}.$$

This recurrence and the initial condition $Z_2(b) = 0$ suggest the definition

$$(4.15) \qquad T_n(b) := -\frac{(n-1)! \, Z_n(b)}{b(1+b)}.$$

Lemma 4.5. The function $T_n(b)$ is a polynomial of degree $n - 3$ with positive integer coefficients.

Proof. The function $T_n(b)$ satisfies the recurrence

$$(4.16) \qquad T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)! \left[\frac{(1+b)^{n-2} - 1}{b}\right].$$

Now simply observe that the right hand side is a polynomial in b. $\qquad\square$

Properties of the polynomial $T_n(b)$ will be described in future publications. We now simply observe that its coefficients are *unimodal*. Recall that a polynomial

$$(4.17) \qquad P_n(b) = \sum_{k=0}^{n} c_k b^k$$

is called *unimodal* if there is an index n^*, such that $c_k \leq c_{k+1}$ for $0 \leq k \leq n^*$ and $c_k \geq c_{k+1}$ for $n^* < k \leq n$. That is, the sequence of coefficients of P_n has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [2] and [6] information about this property. We now use the result of [1] to establish the unimodality of T_n.

Theorem 4.6. Suppose $c_k > 0$ is a nondecreasing sequence. Then $P(x + 1)$ is unimodal.

Therefore we consider the polynomial $S_n(b) := T_n(b - 1)$. It satisfies the recurrence

$$(4.18) \qquad S_n(b) = b(n-2)S_{n-1}(b) + (n-3)! \sum_{r=0}^{n-3} b^r.$$

Now write

$$(4.19) \qquad S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k,$$

and conclude that $c_{0,n} = (n-3)!$ and

$$(4.20) \qquad c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!,$$

from which it follows that

$$(4.21) \qquad c_{k+1,n} - c_{k,n} = (n-2)\left[c_{k,n-1} - c_{k-1,n-1}\right].$$

We conclude that $c_{k,n}$ is a nondecreasing sequence.

Theorem 4.7. The polynomial $T_n(b)$ is unimodal.

Conclusions. We have given explicit formulas for integrals of the form

$$(4.22) \qquad \int_a^b R(x) \, \ln x \, dx,$$

where R is a rational function with real poles. Future reports will describe the case of higher powers

$$(4.23) \qquad \int_a^b R(x) \, \ln^m x \, dx,$$

as well as the case of complex poles, based on integrals of the form

$$(4.24) \qquad C_n(a,r) := \int_0^b \frac{\ln x \, dx}{(x^2 + r^2)^n}.$$

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